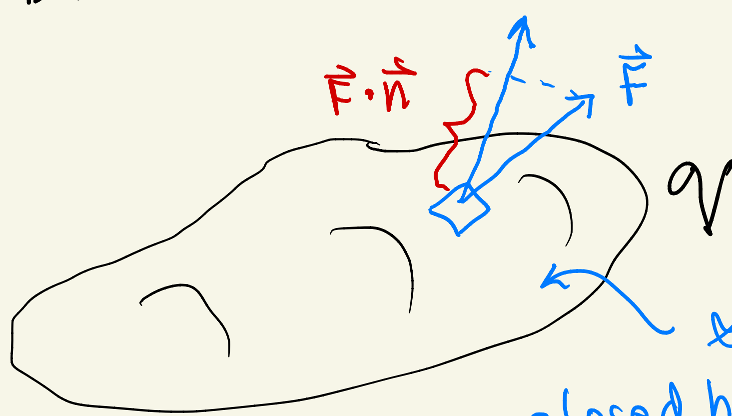
§ 16.9 Divergence Theorem:

Statement:

$$\underbrace{\iiint_{V} \operatorname{Div} \vec{F}\, dV}_{\text{The Ch 15 triple integral of Div } \vec{F} \text{ over a 3-d vol } V} = \underbrace{\iint_{S} \vec{F} \cdot \vec{n}\, dS}_{\text{The flux of } \vec{F} \text{ through the boundary } S \text{ of } V}$$

$\vec{F} = \overrightarrow{(M, N, P)}$ $\quad \operatorname{Div} \vec{F} = M_x + N_y + P_z$

Picture:



In words: "The integral of Div $\vec{F}$ over a volume V is always equal to the flux of $\vec{F}$ through the boundary"

Recall: In the fluid model $\vec{F} = \delta \vec{v}$ = mass flux vector, $\iint_{S} \vec{F} \cdot \vec{n}\, dS = \frac{\text{mass}}{\text{time}}$ outward thru S

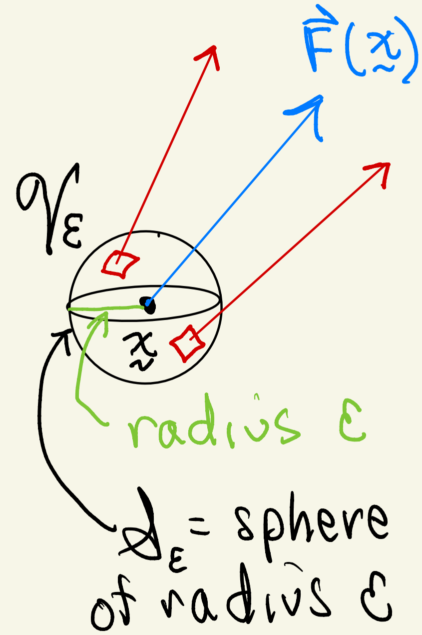
Example: Use the Divergence theorem to give a physical interpretation to $\mathrm{Div}\,\vec{F}$.

That is, we can compute $\mathrm{Div}\,\vec{F}$ at each point $P = \underset{\sim}{x} = (x,y,z)$ as
$$\mathrm{Div}\,\vec{F}(\underset{\sim}{x}) = \underbrace{\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}}_{\text{A number computed at each point } \underset{\sim}{x}}$$

Q: What does $\mathrm{Div}\,\vec{F}$ measure?

Ans: $\mathrm{Div}\,\vec{F}$ measures "Flux per Volume"

To see this, take a small ball V_ε of radius ε centered at $\underset{\sim}{x}$.



Apply the Divergence Thm:

$$\iiint_{V_\varepsilon} \mathrm{Div}\,\vec{F}\, dV = \iint_{S_\varepsilon} \vec{F}\cdot\vec{n}\, dS$$

The trick: If ε is suffiently small, and $\vec{F}$ and its derivatives are continuous, then the value of $\mathrm{Div}\,\vec{F} \approx \mathrm{Div}\,\vec{F}(\underset{\sim}{x})$ throughout V_ε

That is:

$$\iiint_{\mathcal{V}_\varepsilon} \operatorname{Div}\vec{F}\,dv = \iiint_{\mathcal{V}_\varepsilon} \operatorname{Div}\vec{F}(\underset{\sim}{x})\,dV + \text{error}$$

center point

smaller than $|\mathcal{V}_\varepsilon| = \mathrm{Vol}\,\mathcal{V}_\varepsilon$

$$= \operatorname{Div}\vec{F}(\underset{\sim}{x}) \underbrace{\iiint_{\mathcal{V}_\varepsilon} dv}_{|\mathcal{V}_\varepsilon| = \text{vol of } \mathcal{V}_\varepsilon} + \text{error}$$

$$= |\mathcal{V}_\varepsilon| \operatorname{Div}\vec{F}(\underset{\sim}{x}) + \text{error}.$$

Solving for $\operatorname{Div}\vec{F}(\underset{\sim}{x})$ gives:

$$\operatorname{Div}\vec{F}(\underset{\sim}{x}) = \frac{1}{|\mathcal{V}_\varepsilon|} \iint_{\mathcal{S}_\varepsilon} \vec{F}\cdot\vec{n}\,dS + \underbrace{\frac{\text{error}}{|\mathcal{V}_\varepsilon|}}_{\text{tends to } 0 \text{ as } \varepsilon \to 0}$$

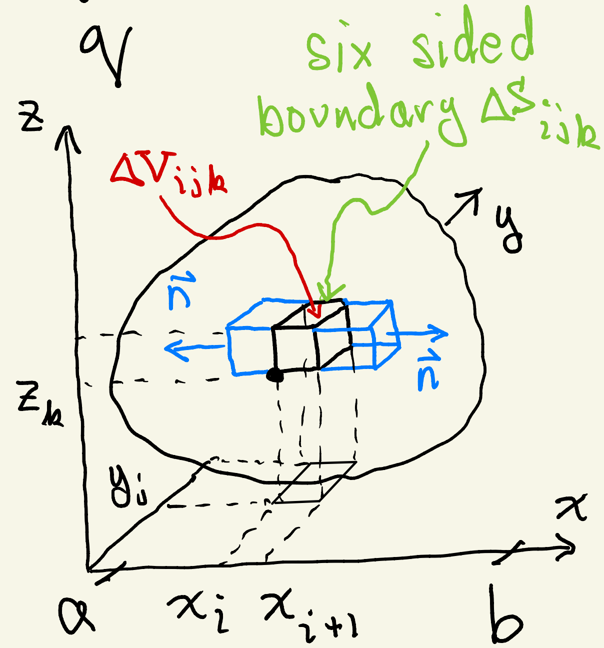
Conclude: $\operatorname{Div}\vec{F}(\underset{\sim}{x}) = \lim_{\varepsilon \to 0} \frac{1}{|\mathcal{V}_\varepsilon|} \iint_{\mathcal{S}_\varepsilon} \vec{F}\cdot\vec{n}\,dS$

$\doteq$ "Flux of $\vec{F}$ per Volume"

Note: This works for any volume $\mathcal{V}_\varepsilon = \underbrace{\varepsilon \mathcal{V}_0}_{\text{so it scales w } \varepsilon}$

Example: The fact that $\text{Div}\vec{F} = \frac{\text{Flux}}{\text{vol}}$ explains why the Divergence Theorem is true:

I.e. Approximate triple integral $\iiint_{V} \text{Div}\vec{F}\, dV$ as a Riemann Sum:



$$\iiint_{V} \text{Div}\vec{F}\, dV \approx \sum_{i,j,k} \text{Div}\vec{F}_{ijk}\, \Delta V$$

$|\Delta V_{ijk}|$

$$\approx \sum_{ijk} \frac{\iint_{\Delta S_{ijk}} \vec{F}\cdot\vec{n}\, dS}{\cancel{\Delta V}} \cancel{\Delta V}$$

$$= \sum_{ijk} \iint_{\Delta S_{ijk}} \vec{F}\cdot\vec{n}\, dS$$

The flux integral cancels on all shared sides as outer normal switches sign!

$$\approx \iint_{S} \vec{F}\cdot\vec{n}\, dS$$

only the outer sides on the boundary of V have no adjacent boundary to cancel them!

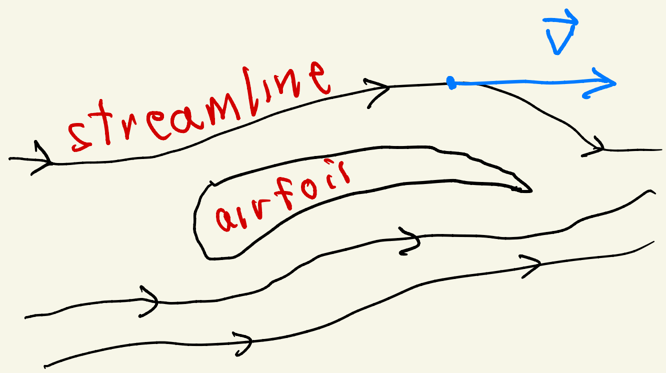
Example: The most important application of the Divergence Theorem –

Assume $\vec{F} = \delta \vec{v}$ in the fluid model where

$\delta(x,y,z) = \dfrac{\text{mass}}{\text{vol}}$

and

$\vec{v}(x,y,z) = \text{velocity}$



Q: What constraint must δ and $\vec{v}$ satisfy to ensure that mass is conserved?

Ans: $\boxed{\mathrm{Div}\, \delta \vec{v} = 0}$ Intuitively, $\mathrm{Div}\, \delta \vec{v}(\underset{\sim}{x}) = \dfrac{\text{Flux}}{\text{vol}}$ at $\underset{\sim}{x}$, so if $\mathrm{Div}\, \delta \vec{v}(\underset{\sim}{x}) \neq 0$, mass is either created or destroyed at $\underset{\sim}{x}$. We get a careful argument by Div Thm

The idea – if mass is conserved in every volume $\mathcal{V}$, then we must have that flux of mass thru the boundary of $\mathcal{V}$ must $= 0$ ... otherwise mass would be accumulating in $\mathcal{V}$.

So Conservation of Mass $\Rightarrow \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS = 0$ for all $\mathcal{V}$

Defn: We say conservation of mass holds if $\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,dS = 0$ for $\mathcal{S}$ the boundary of any 3-d Surface $\mathcal{V}$. Now apply Div. Thm

Div Thm: $\iiint_{\mathcal{V}} \mathrm{Div}\,\vec{F}\,dV = \iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,dS = 0$

This implies $\mathrm{Div}\,\vec{F} = 0$. If $\mathrm{div}\,\vec{F}(\underset{\sim}{x}) \neq 0$ at a point, then choose a small volume $\mathcal{V}_\varepsilon$ in which $\mathrm{div}\,\vec{F}$ has same sign as $\mathrm{Div}\,\vec{F}(\underset{\sim}{x})$, say positive, then

$$\iiint_{\mathcal{V}_\varepsilon} \underbrace{\mathrm{Div}\,\vec{F}}_{\text{pos}}\,d\vec{V} > 0 \quad \#$$

Conclude: the condition for Conservation of Mass is:

$$\boxed{\mathrm{Div}\,\delta\vec{v} = 0}$$

Continuity Equation

- For time dependent flows, same idea

$$\delta(x,y,z,t)\ ,\ \vec{v}(x,y,z,t)$$

Then Conservation of mass is

$$\underbrace{\delta_t + \mathrm{Div}(\delta\vec{v}) = 0}_{\mathrm{Div}_{t,\underset{\sim}{x}}(\delta,\ \delta\vec{v}) = 0} \qquad \text{Continuity Equation} \qquad (*)$$

The continuity equation is the first partial differential equation of fluid mechanics. For example, the continuum version of Newton's Laws expresses conservation of mass (*) together with an equation for conservation of momentum and conservation of energy, both of which can be derived from the Divergence Theorem. Note that (*) is NONLINEAR! The equations for Fluids are "Fiercely Nonlinear"